THE UNIVERSAL CENTRAL EXTENSION OF THE THREE-POINT $\mathfrak{sl}_2$ LOOP ALGEBRA

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Abstract. We consider the three-point loop algebra,

$$L = \mathfrak{sl}_2 \otimes \mathbb{F}[t, t^{-1}, (t-1)^{-1}],$$

where $\mathbb{F}$ denotes a field of characteristic 0 and t is an indeterminate. The universal central extension $\widehat{L}$ of L was determined by Bremner. In this note, we give a presentation for $\widehat{L}$ via generators and relations, which highlights a certain symmetry over the alternating group A_4. To obtain our presentation of $\widehat{L}$, we use the realization of L as the tetrahedron Lie algebra.

1. Introduction

Throughout this paper $\mathbb{F}$ will denote a field of characteristic 0. Consider the Lie algebra $\mathfrak{sl}_2$ over $\mathbb{F}$, and for an indeterminate t, the polynomial algebra $\mathbb{F}[t]$ localized at t and $t-1$:

$$\mathcal{A} = \mathbb{F}[t, t^{-1}, (t-1)^{-1}].$$

(Localization at any two $t-\alpha_1$, $t-\alpha_2$ for distinct $\alpha_1, \alpha_2 \in \mathbb{F}$ results in an algebra isomorphic to $\mathcal{A}$, so there is no loss of generality in assuming one value is 0 and the other is 1.) The *loop algebra* corresponding to $\mathfrak{sl}_2$ and $\mathcal{A}$ is the Lie algebra

$$L = \mathfrak{sl}_2 \otimes \mathcal{A},$$

with product $[x \otimes a, y \otimes b] = [x, y] \otimes ab$.

Our primary focus here is on central extensions of L, so we begin by recalling a few relevant definitions. A *central extension* of a Lie algebra $\mathcal{L}$ is a pair $(\mathcal{K}, \pi)$ consisting of a Lie algebra $\mathcal{K}$ and a surjective Lie algebra homomorphism $\pi : \mathcal{K} \to \mathcal{L}$ whose kernel lies in the center of $\mathcal{K}$. Given central extensions $(\mathcal{K}, \pi)$ and $(\mathcal{K}', \pi')$ of $\mathcal{L}$, by a *homomorphism* (resp. *isomorphism*) from $(\mathcal{K}, \pi)$ to $(\mathcal{K}', \pi')$ we mean a homomorphism (resp. isomorphism) of Lie algebras $\varphi : \mathcal{K} \to \mathcal{K}'$ such that $\pi = \pi' \circ \varphi$.

A central extension $(\mathcal{K}, \pi)$ of $\mathcal{L}$ is *universal* whenever there exists a homomorphism from $(\mathcal{K}, \pi)$ to any other central extension of $\mathcal{L}$. A Lie algebra

Date: December 17, 2005.

*Support from NSF grant #DMS–0245082 is gratefully acknowledged.

Keywords. $\mathfrak{sl}_2$ loop algebra, universal central extension, tetrahedron Lie algebra, Onsager Lie algebra.

2000 Mathematics Subject Classification. 17B37.

$\mathcal{L}$ possesses a universal central extension if and only if $\mathcal{L}$ is perfect (i.e. $\mathcal{L} = [\mathcal{L}, \mathcal{L}]$), and in this case, the universal central extension of $\mathcal{L}$ is unique up to isomorphism. It is well-established that the universal central extension plays a crucial role in representation theory; one need only look at the examples of the affine, Virasoro, and toroidal Lie algebras for affirmation of this statement (see [Ka], [MP]).

The loop algebra $L = \mathfrak{sl}_2 \otimes \mathcal{A}$ is easily seen to be perfect; therefore, L has a universal central extension which we denote by $(\widehat{L}, \pi)$. Bremner [Br] has given a detailed description of $(\widehat{L}, \pi)$. He has shown that the center of $\widehat{L}$ has dimension 2, and he has given an explicit basis and Lie bracket for $\widehat{L}$.

Our goal here is to give a presentation for $\widehat{L}$ via generators and relations that highlights a certain symmetry over the alternating group A_4.

Our point of departure is the realization of L as the tetrahedron algebra given by Hartwig and Terwilliger.

Definition 1.1. [HT] Let $\boxtimes$ denote the Lie algebra over $\mathbb{F}$ defined by generators $\{x_{i,j} \mid i, j \in \mathbb{I}, i \neq j\}$, $\mathbb{I} = \{0, 1, 2, 3\}$, and the following relations:

(i) For distinct $i, j \in \mathbb{I}$,

$$x_{i,j} + x_{j,i} = 0. \tag{1.2}$$

(ii) For mutually distinct $i, j, k \in \mathbb{I}$,

$$[x_{i,j}, x_{j,k}] = 2x_{i,j} + 2x_{j,k}. \tag{1.3}$$

(iii) For mutually distinct $i, j, k, \ell \in \mathbb{I}$,

$$[x_{i,j}[x_{i,j}, [x_{i,j}, x_{k,\ell}]\,]\,] = 4[x_{i,j}, x_{k,\ell}]. \tag{1.4}$$

We call $\boxtimes$ the *tetrahedron algebra*.

Theorem 1.5. [HT] *The Lie algebras $\boxtimes$ and L are isomorphic.*

We will obtain our presentation of $\widehat{L}$ as follows. First we will display a central extension $(\widehat{\boxtimes}, \pi)$ of $\boxtimes$, with $\widehat{\boxtimes}$ defined by generators and relations. Then we will modify the Lie algebra isomorphism $\sigma : \boxtimes \to L$ given in [HT] to obtain a homomorphism of Lie algebras, $\widehat{\sigma} : \widehat{\boxtimes} \to \widehat{L}$, such that the following diagram commutes:

$$\begin{tikzcd}
\widehat{\boxtimes} \arrow[r, "\pi"] \arrow[d, "\widehat{\sigma}"'] & \boxtimes \arrow[d, "\sigma"] \\
\widehat{L} \arrow[r, "\pi"] & L
\end{tikzcd} \tag{1.6}$$

Using this and the universality of $(\widehat{L}, \pi)$, we will argue that $\widehat{\sigma} : \widehat{\boxtimes} \to \widehat{L}$ is an isomorphism and thereby determine a presentation of $\widehat{L}$ by generators and relations.

2. The isomorphism $\sigma : \boxtimes \to L$

Let $\{e, f, h\}$ denote the canonical basis for $\mathfrak{sl}_2$ having product $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$. The basis

$$X = 2e - h, \qquad Y = -2f - h, \qquad Z = h$$

is more suitable for our purposes. Following [ITW] we call X, Y, Z the *equitable basis* for $\mathfrak{sl}_2$, since

$$[X, Y] = 2X + 2Y, \qquad [Y, Z] = 2Y + 2Z, \qquad [Z, X] = 2Z + 2X.$$

There exists a unique $\mathbb{F}$-algebra automorphism $\prime$ of $\mathcal{A}$ such that $t' = 1-t^{-1}$. This automorphism has order 3 and satisfies

$$t'' = (1 - t)^{-1}, \qquad tt' = t - 1, \tag{2.1}$$

$$t't'' = t' - 1, \qquad t''t = t'' - 1, \tag{2.2}$$

where $t'' = (t')'$. The relations in (2.1) and (2.2) imply that the following is a basis for the $\mathbb{F}$-vector space $\mathcal{A}$:

$$\{1\} \cup \{t^i, (t')^i, (t'')^i \mid i \in \mathbb{N}\},$$

where $\mathbb{N} = \{1, 2, 3, \dots\}$.

Proposition 2.3. ([HT, Thm. 11.5]) *There exists an isomorphism of Lie algebras $\sigma : \boxtimes \to L$ that sends*

$$\begin{aligned} x_{1,2} &\to X \otimes 1, & x_{0,3} &\to Y \otimes t + Z \otimes (t - 1), \\ x_{2,3} &\to Y \otimes 1, & x_{0,1} &\to Z \otimes t' + X \otimes (t' - 1), \\ x_{3,1} &\to Z \otimes 1, & x_{0,2} &\to X \otimes t'' + Y \otimes (t'' - 1) \end{aligned}$$

where X, Y, Z is the equitable basis for $\mathfrak{sl}_2$.

3. A central extension of $\boxtimes$

In this section, we will construct a central extension $(\widehat{\boxtimes}, \pi)$ of $\boxtimes$. Later we will show that this extension is universal. From now on, we identify the symmetric group S_4 with the group of permutations of $\mathbb{I}$.

Definition 3.1. Given a sequence (i, j, k) of mutually distinct elements of $\mathbb{I}$, there exists a unique $\tau \in S_4$ such that $\tau(0) = i$, $\tau(1) = j$, and $\tau(2) = k$. The sequence (i, j, k) is said to be *even* (resp. *odd*) whenever $\tau \in A_4$ (resp. $\tau \notin A_4$), where A_4 is the alternating subgroup of S_4.

Definition 3.2. A partition p of $\mathbb{I}$ into two (disjoint) subsets, each with two elements, is said to have *shape* $(2, 2)$.

The set $P(\mathbb{I})$ of all partitions of $\mathbb{I}$ of shape $(2, 2)$ has cardinality 3.

Definition 3.3. Let $\widehat{\boxtimes}$ denote the Lie algebra over $\mathbb{F}$ defined by generators

$$\{X_{i,j} \mid i,j \in \mathbb{I}, i \neq j\} \cup \{C_p \mid p \in P(\mathbb{I})\}$$

and the following relations:

(i) For $p \in P(\mathbb{I})$,

$$C_p \;\text{ is central.}$$

(ii)

$$\sum_{p \in P(\mathbb{I})} C_p = 0.$$

(iii) For distinct $i, j \in \mathbb{I}$,

$$X_{i,j} + X_{j,i} = C_p,$$

where $p \in P(\mathbb{I})$ consists of $\{i, j\}$ and its complement in $\mathbb{I}$.

(iv) For mutually distinct $i, j, k \in \mathbb{I}$ such that (i, j, k) is even,

$$[X_{i,j}, X_{j,k}] = 2X_{i,j} + 2X_{j,k}.$$

(v) For mutually distinct $i, j, k, \ell \in \mathbb{I}$,

$$[X_{i,j}[X_{i,j}, [X_{i,j}, X_{k,\ell}]\,]\,] = 4[X_{i,j}, X_{k,\ell}].$$

Lemma 3.4. *There exists a surjective homomorphism of Lie algebras $\pi : \widehat{\boxtimes} \to \boxtimes$ such that*

$$\begin{aligned} \pi(X_{i,j}) &= x_{i,j} && i, j \in \mathbb{I},\ i \neq j, \\ \pi(C_p) &= 0 && p \in P(\mathbb{I}). \end{aligned}$$

Proof: Compare the defining relations for $\boxtimes$ and $\widehat{\boxtimes}$ given in Definitions 1.1 and 3.3. $\square$

Lemma 3.5. *For mutually distinct $i, j, k \in \mathbb{I}$ such that (i, j, k) is odd, in the algebra $\widehat{\boxtimes}$ we have*

$$[X_{i,j}, X_{j,k}] = 2X_{i,j} + 2X_{j,k} + 2C_p, \tag{3.6}$$

where $p \in P(\mathbb{I})$ consists of $\{i, k\}$ and its complement in $\mathbb{I}$.

Proof: The sequence (k, j, i) is even since (i, j, k) is odd. Therefore, by Definition 3.3 (iv),

$$[X_{k,j}, X_{j,i}] = 2X_{k,j} + 2X_{j,i}.$$

Evaluating this using (i)-(iii) of Definition 3.3, we obtain (3.6). $\square$

Lemma 3.7. *The following subspaces of $\widehat{\boxtimes}$ coincide:*

(i) *the kernel of π,*

(ii) $\mathrm{Span}\{C_p \mid p \in P(\mathbb{I})\}$,

(iii) *the center of $\widehat{\boxtimes}$.*

Proof: Set $\mathcal{C} = \mathrm{Span}\{C_p \mid p \in P(\mathbb{I})\}$. We first show $\mathcal{C} = \mathrm{Ker}(\pi)$. We have $\mathcal{C} \subseteq \mathrm{Ker}(\pi)$ by Lemma 3.4. To establish equality, observe that $\mathcal{C}$ is an ideal in $\widehat{\boxtimes}$, and let $\pi' : \widehat{\boxtimes} \to \widehat{\boxtimes}/\mathcal{C}$ denote canonical surjection with kernel $\mathcal{C}$. Since $\pi'(C_p) = 0$ for $p \in P(\mathbb{I})$, it follows from Definition 3.3 and Lemma 3.5 that the elements $\{\pi'(X_{i,j}) \mid i, j \in \mathbb{I},\ i \neq j\}$ satisfy the defining relations (1.2)–(1.4) for $\boxtimes$. Therefore, there exists a Lie algebra homomorphism $\gamma : \boxtimes \to \widehat{\boxtimes}/\mathcal{C}$ such that $\gamma(x_{i,j}) = \pi'(X_{i,j})$ for all distinct $i, j \in \mathbb{I}$. From the construction, the following diagram commutes:

$$\begin{tikzcd} \widehat{\boxtimes} \arrow[r, "\pi"] \arrow[d, "\mathrm{id}"'] & \boxtimes \arrow[d, "\gamma"] \\ \widehat{\boxtimes} \arrow[r, "\pi'"] & \widehat{\boxtimes}/\mathcal{C} \end{tikzcd}$$

We may now argue

$$\begin{aligned} \mathcal{C} &= \mathrm{Ker}(\pi') \\ &= \mathrm{Ker}(\gamma \circ \pi) \\ &\supseteq \mathrm{Ker}(\pi), \end{aligned}$$

which implies that $\mathcal{C} = \mathrm{Ker}(\pi)$. Next we prove that the center $Z(\widehat{\boxtimes}) = \mathcal{C}$. We have $\mathcal{C} \subseteq Z(\widehat{\boxtimes})$ by Definition 3.3 (i). To obtain the reverse inclusion, it suffices to show that the image of $Z(\widehat{\boxtimes})$ under π is zero. This image is contained in $Z(\boxtimes)$ since π is surjective. But $\boxtimes$ is isomorphic to L and $Z(L) = 0$, so $Z(\boxtimes) = 0$. Therefore the image of $Z(\widehat{\boxtimes})$ under π is zero and consequently $Z(\widehat{\boxtimes}) \subseteq \mathcal{C}$. From these comments, we find that $Z(\widehat{\boxtimes}) = \mathcal{C}$. □

Corollary 3.8. *The pair* $(\widehat{\boxtimes}, \pi)$ *is a central extension of* $\boxtimes$.

Proof: The map $\pi : \widehat{\boxtimes} \to \boxtimes$ is a surjective homomorphism of Lie algebras whose kernel is contained in the center of $\widehat{\boxtimes}$. □

4. The universal central extension of $L = \mathfrak{sl}_2 \otimes \mathcal{A}$

In this section, we give a detailed description of the universal central extension $(\widehat{L}, \pi)$ of L. In the next section we will use this description to define the map $\widehat{\sigma} : \widehat{\boxtimes} \to \widehat{L}$ mentioned in Section 1.

By results of Kassel [K] (see also [KL] and [BK]), the universal central extension of $\mathcal{L} := \mathfrak{g} \otimes \mathcal{B}$ for any finite-dimensional complex simple Lie algebra $\mathfrak{g}$ and any commutative, associative algebra $\mathcal{B}$ with 1 is obtained from $\mathcal{L}$ by adjoining the Kähler differentials modulo the exact forms of $\mathcal{B}$. This description enabled Bremner [Br] to show that the universal central extension of any n-point loop algebra over $\mathfrak{g}$ has an $(n-1)$-dimensional kernel. The same argument works over any field $\mathbb{F}$ of characteristic 0. Applying this result to our loop algebra $L = \mathfrak{sl}_2 \otimes \mathcal{A}$, we see that $\dim_{\mathbb{F}} \widehat{L}/L = 2$.

An alternative description of $\widehat{L}$ can be found in [ABG]. Let $\mathcal{S}$ denote the subspace of $\mathcal{A}\otimes\mathcal{A}$ spanned by the elements $a\otimes b+b\otimes a$ and $ab\otimes c+bc\otimes a+ca\otimes b$ for all $a, b, c \in \mathcal{A}$. Let $\langle \mathcal{A}, \mathcal{A}\rangle = (\mathcal{A}\otimes\mathcal{A})/\mathcal{S}$, and write $\langle a, b\rangle = (a\otimes b)+\mathcal{S}$. From the construction we know that

$$\langle a, b\rangle + \langle b, a\rangle = 0, \tag{4.1}$$

$$\langle ab, c\rangle + \langle bc, a\rangle + \langle ca, b\rangle = 0 \tag{4.2}$$

for all $a, b, c \in \mathcal{A}$. Then

$$\widehat{L} = \big(\mathfrak{sl}_2 \otimes \mathcal{A}\big) \oplus \langle \mathcal{A}, \mathcal{A}\rangle,$$

where $\langle \mathcal{A}, \mathcal{A}\rangle$ is central and

$$[x\otimes a, y\otimes b] = [x, y]\otimes ab + (x\,|\,y)\langle a, b\rangle$$

for all $x, y \in \mathfrak{sl}_2$ and $a, b \in \mathcal{A}$. Here $(x\,|\,y)$ denotes the Killing form of $\mathfrak{sl}_2$. Thus, finding $\widehat{L}$ amounts to computing $\langle \mathcal{A}, \mathcal{A}\rangle$ explicitly.

Using the relations (4.1), (4.2), it is not difficult to show, just as in the affine case (see [Ka], for example), that

$$\langle f^m, f^n\rangle = m\delta_{m+n,0}\langle f, f^{-1}\rangle \tag{4.3}$$

for $f = t, t'$, or t'' and all integers m, n, where δ is the Kronecker delta. Letting $g = f'$ and using $f' = 1 - f^{-1}$, we have

$$\begin{aligned} \langle f^m, g^n\rangle &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\langle f^m, f^{-k}\rangle \\ &= m(-1)^m\binom{n}{m}\langle f, f^{-1}\rangle \end{aligned} \tag{4.4}$$

for nonnegative integers m, n. From (2.2) we find that $tt't'' = -1$; using this and (4.2) we obtain

$$\begin{aligned} \langle t'', (t'')^{-1}\rangle &= -\langle t'', tt'\rangle \\ &= \langle t, t't''\rangle + \langle t', t''t\rangle \\ &= -\langle t, t^{-1}\rangle - \langle t', (t')^{-1}\rangle. \end{aligned}$$

It follows from these computations that $\langle \mathcal{A}, \mathcal{A}\rangle$ is spanned by $\langle t, t^{-1}\rangle$ and $\langle t', (t')^{-1}\rangle$ and that

$$\langle t, t^{-1}\rangle + \langle t', (t')^{-1}\rangle + \langle t'', (t'')^{-1}\rangle = 0.$$

Since $\widehat{L}/L$ has dimension 2, the space $\langle \mathcal{A}, \mathcal{A}\rangle$ has dimension 2. Consequently $\langle t, t^{-1}\rangle$ and $\langle t', (t')^{-1}\rangle$ form a basis for $\langle \mathcal{A}, \mathcal{A}\rangle$.

The next result is now apparent.

Theorem 4.5. ([Br], [ABG]) *Let C denote a two-dimensional vector space over $\mathbb{F}$. Let $\mathfrak{c}, \mathfrak{c}'$ denote a basis for C and define $\mathfrak{c}''$ so that*

$$\mathfrak{c} + \mathfrak{c}' + \mathfrak{c}'' = 0.$$

Then the following (i)–(iii) *hold.*

(i) *There exists a Lie algebra*

$$\widehat{L} = L \oplus C$$

with product

$$[\widehat{L}, C] = 0,$$
$$[x \otimes a, y \otimes b] = [x, y] \otimes ab + (x \,|\, y)\langle a, b\rangle$$

for $x, y \in \mathfrak{sl}_2$ *and* $a, b \in \mathcal{A}$*, where* $(x \,|\, y)$ *is the Killing form for* $\mathfrak{sl}_2$*, and where* $\langle\, ,\, \rangle : \mathcal{A} \times \mathcal{A} \to C$ *is* $\mathbb{F}$*-bilinear and satisfies*

$\langle\, ,\, \rangle$	*1*	t^j	$(t')^j$	$(t'')^j$
1	0	0	0	0
t^i	0	0	$(-1)^i i \binom{j}{i} \mathfrak{c}$	$(-1)^{j+1} j \binom{i}{j} \mathfrak{c}''$
$(t')^i$	0	$(-1)^{j+1} j \binom{i}{j} \mathfrak{c}$	0	$(-1)^i i \binom{j}{i} \mathfrak{c}'$
$(t'')^i$	0	$(-1)^i i \binom{j}{i} \mathfrak{c}''$	$(-1)^{j+1} j \binom{i}{j} \mathfrak{c}'$	0

for $i, j \in \mathbb{N}$.

(ii) *There exists a homomorphism of Lie algebras* $\pi : \widehat{L} \to L$ *that has kernel* C *and acts as the identity on* L.

(iii) *The pair* $(\widehat{L}, \pi)$ *is the universal central extension of* L.

5. A HOMOMORPHISM $\widehat{\sigma} : \widehat{\boxtimes} \to \widehat{L}$

Lemma 5.1. *There exists a unique Lie algebra homomorphism* $\widehat{\sigma} : \widehat{\boxtimes} \to \widehat{L}$ *specified by the following tables.*

p	Image of C_p under $\widehat{\sigma}$
$\{0,1\}\{2,3\}$	$-4\mathfrak{c}''$
$\{0,2\}\{1,3\}$	$-4\mathfrak{c}$
$\{0,3\}\{1,2\}$	$-4\mathfrak{c}'$

i	j	Image of $X_{i,j}$ under $\widehat{\sigma}$
1	2	$X \otimes 1 - 4\mathfrak{c}'$
2	1	$-X \otimes 1$
2	3	$Y \otimes 1 - 4\mathfrak{c}''$
3	2	$-Y \otimes 1$
3	1	$Z \otimes 1 - 4\mathfrak{c}$
1	3	$-Z \otimes 1$
0	3	$Y \otimes t + Z \otimes (t-1) + 4\mathfrak{c}$
3	0	$-Y \otimes t - Z \otimes (t-1) + 4\mathfrak{c}''$
0	1	$Z \otimes t' + X \otimes (t'-1) + 4\mathfrak{c}'$
1	0	$-Z \otimes t' - X \otimes (t'-1) + 4\mathfrak{c}$
0	2	$X \otimes t'' + Y \otimes (t''-1) + 4\mathfrak{c}''$
2	0	$-X \otimes t'' - Y \otimes (t''-1) + 4\mathfrak{c}'$

,

where X, Y, Z is the equitable basis for $\mathfrak{sl}_2$.

Proof: It is routine to verify that the elements in the above tables satisfy the defining relations for $\widehat{\boxtimes}$ given in Definition 3.3. $\square$

Lemma 5.2. *The diagram in (1.6) commutes.*

Proof: By Definition 3.3, the following is a generating set for $\widehat{\boxtimes}$:

$$\{X_{i,j} \mid i, j \in \mathbb{I}, i \neq j\} \cup \{C_p \mid p \in P(\mathbb{I})\}.$$

We chase these generators around the diagram using the maps in Proposition 2.3, Lemma 3.4, Theorem 4.5(ii), and Lemma 5.1. For each generator the image under the composition $\pi \circ \widehat{\sigma}$ coincides with the image under the composition $\sigma \circ \pi$. $\square$

Theorem 5.3. *The Lie algebra homomorphism $\widehat{\sigma} : \widehat{\boxtimes} \to \widehat{L}$ from Lemma 5.1 is an isomorphism.*

Proof: We first show that $\widehat{\sigma}$ is surjective. The map $\pi : \widehat{\boxtimes} \to \boxtimes$ is surjective, and the map $\sigma : \boxtimes \to L$ is an isomorphism, so the composite map $\sigma \circ \pi : \widehat{\boxtimes} \to L$ is surjective. Therefore by Lemma 5.2, the composition $\pi \circ \widehat{\sigma} : \widehat{\boxtimes} \to L$ is surjective. The kernel of $\pi : \widehat{L} \to L$ is the space C from Theorem 4.5, and C is contained in the image of $\widehat{\sigma}$, so $\widehat{\sigma}$ is surjective. We now argue that $\widehat{\sigma}$ is injective. As before, set $\mathcal{C} = \text{Span}\{C_p \mid p \in P(\mathbb{I})\}$. The map $\pi : \widehat{\boxtimes} \to \boxtimes$ has kernel $\mathcal{C}$ by Lemma 3.7, and the map $\sigma : \boxtimes \to L$ is an isomorphism, so the composition $\sigma \circ \pi : \widehat{\boxtimes} \to L$ has kernel $\mathcal{C}$. By this and Lemma 5.2, the composition $\pi \circ \widehat{\sigma} : \widehat{\boxtimes} \to L$ has kernel $\mathcal{C}$. Consequently, the kernel of $\widehat{\sigma}$ is contained in $\mathcal{C}$. From the first table of Lemma 5.1, and the fact that $\mathfrak{c}, \mathfrak{c}'$ form a basis for C, we find that the restriction of $\widehat{\sigma}$ to $\mathcal{C}$ is injective. Therefore $\widehat{\sigma}$ is injective and hence an isomorphism. $\square$

Corollary 5.4. *The central extension $(\widehat{\boxtimes}, \pi)$ of $\boxtimes$ given in Corollary 3.8 is universal.*

Proof: By Lemma 5.2, the diagram in (1.6) commutes. The map σ is an isomorphism by Proposition 2.3, and $\widehat{\sigma}$ is an isomorphism by Theorem 5.3. The result follows from this, since $(\widehat{L}, \pi)$ is the universal central extension of L. □

6. The defining relations for $\widehat{\boxtimes}$ revisited

Definition 3.3 gives the defining relations for the Lie algebra $\widehat{\boxtimes}$. In this section, we re-express these relations, this time using a notation that makes explicit the role of the alternating group A_4.

We will view S_4 as acting on $\mathbb{I}$ from the right; this means that when applying a product $\alpha\beta$, we first apply α and then β. We consider the following normal subgroup of S_4:

$$N \quad = \quad \{(01)(23),\ (02)(31),\ (03)(12),\ e\}.$$

This subgroup is contained in A_4 and is therefore a normal subgroup of A_4. Let N' denote the set of nonidentity elements of N. For each $\eta \in N'$ let $[\eta]$ denote the partition of $\mathbb{I}$ consisting of the orbits of η. Note that the map $\eta \to [\eta]$ is a bijection from N' to $P(\mathbb{I})$. Two elements of A_4,

$$\zeta = (01)(23), \qquad\qquad \vartheta = (012),$$

play a distinguished role in our computations. The first belongs to N', while the second one does not. Together they generate A_4.

Theorem 6.1. *$\widehat{\boxtimes}$ is isomorphic to the Lie algebra over $\mathbb{F}$ that has generators*

$$X_\alpha, \quad C_\eta \qquad\qquad \alpha \in A_4, \quad \eta \in N'$$

and the following relations:

(i) *For $\eta \in N'$,*

$$C_\eta \quad \text{is central.}$$

(ii)

$$\sum_{\eta \in N'} C_\eta = 0.$$

(iii) *For $\alpha \in A_4$,*

$$X_\alpha + X_{\zeta\alpha} = C_{\alpha^{-1}\zeta\alpha}.$$

(iv) *For $\alpha \in A_4$,*

$$[X_\alpha, X_{\vartheta\alpha}] = 2X_\alpha + 2X_{\vartheta\alpha}.$$

(v) *For $\alpha \in A_4$ and for $\eta \in N'$, $\eta \neq \zeta$,*

$$[X_\alpha, [X_\alpha, [X_\alpha, X_{\eta\alpha}]]] = 4[X_\alpha, X_{\eta\alpha}].$$

An isomorphism with the presentation in Definition 3.3 is given by

$$\begin{array}{rclr} X_\alpha & \to & X_{\alpha(0),\alpha(1)} & \alpha \in A_4 \\ C_\eta & \to & C_{[\eta]} & \eta \in N'. \end{array}$$

Proof: Up to notation, the above presentation of $\widehat{\boxtimes}$ is the same as the one given in Definition 3.3. □

7. Concluding Remarks

We conclude with some comments relating our results to the Onsager Lie algebra. This algebra was introduced in a seminal paper [O] in which the free energy of the two-dimensional Ising model was computed exactly. Since then it has been widely investigated by both the physics and mathematics communities in connection with solvable lattice models, representation theory, Kac-Moody Lie algebras, tridiagonal pairs, and partially orthogonal polynomials. In [P], Perk showed that the Onsager Lie algebra has a presentation by generators A, B and the following relations:

$$\begin{array}{rcl} [A,[A,[A,B]]] & = & 4[A,B] \\ [B,[B,[B,A]]] & = & 4[B,A]. \end{array}$$

Let Ω (resp. Ω') (resp. Ω'') denote the subalgebra of $\boxtimes$ generated by $x_{0,1}$ and $x_{2,3}$ (resp. $x_{0,2}$ and $x_{1,3}$) (resp. $x_{0,3}$ and $x_{1,2}$). It was shown in [HT] that each of the Lie algebras Ω, Ω', Ω'' is isomorphic to the Onsager algebra, and that $\boxtimes$ is their direct sum. Our results lead to a similar decomposition of $\widehat{\boxtimes}$ as follows: Let $\mathcal{O}$ (resp. $\mathcal{O}'$) (resp. $\mathcal{O}''$) denote the subalgebra of $\widehat{\boxtimes}$ generated by $X_{0,1}$ and $X_{2,3}$ (resp. $X_{0,2}$ and $X_{1,3}$) (resp. $X_{0,3}$ and $X_{1,2}$). Using the commuting diagram (1.6) and the tables in Lemma 5.1, we find that each of the algebras $\mathcal{O}$, $\mathcal{O}'$, $\mathcal{O}''$ is isomorphic to the Onsager algebra, and that $\widehat{\boxtimes}$ is the direct sum $\mathcal{O} + \mathcal{O}' + \mathcal{O}'' + \mathcal{C}$, where $\mathcal{C} := \mathrm{Span}\{C_p \mid p \in P(\mathbb{I})\}$ is the two-dimensional center of $\widehat{\boxtimes}$. Using the isomorphism $\widehat{\sigma} : \widehat{\boxtimes} \to \widehat{L}$ in Theorem 5.3, we obtain a corresponding decomposition for the universal central extension $\widehat{L}$ of the loop algebra L.

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